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First-Order Linear PDEs and Uniqueness in the Cauchy Problem

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INTRODUCTION AND STATEMENT OF RESULTS

Let Ω be an open subset of $\mathbb{R}^N$, $N \geqslant 2$, where the variable will momentarily be denoted by $y = (y^1,..., y^N)$. We shall deal with a first-order linear partial differential operator

$$L = \sum_{j=1}^{N} c^j(y) \frac{\partial}{\partial y^j} + c^0(y), \tag{1}$$

whose coefficients c^j $(0 \leqslant j \leqslant N)$ are complex C^∞ functions in Ω. We assume that we are given a C^∞ function, real-valued, $\Phi(y)$, in Ω, whose differential $d\Phi$ is noncharacteristic with respect to L at a point y^0 of Ω:

$$\sum_{j=1}^{N} c^j(y^0) \frac{\partial \Phi}{\partial y^j}(y^0) \neq 0. \tag{2}$$

The class of operators (1) with which we shall be mainly concerned will satisfy the *solvability condition* (**P**) of [2], at the point y^0. Many statements of the latter are possible (see [2, 3]). We may select the following: because of (2) we can always find a complex number z such that

$$zL = X + iY, \qquad i = (-1)^{1/2},$$

where X and Y are two real vector fields in Ω, such that y^0 is *not* a critical

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point of X (in fact, at least one of the two values, $z = 1$ or $z = i$, suffices). Then one says that L satisfies **(P)** at y^0 if:

(P) *there is an open neighborhood $\mathscr{V}$ of y^0 such that the vector field Y does not change direction along any characteristic curve of X contained in $\mathscr{V}$.*

The main result of the present paper can then be stated as follows.

THEOREM 1. *Suppose that* **(P)** *and* (2) *hold. To every open neighborhood $\mathscr{U} \subset \Omega$ of y^0 there is another open neighborhood $\mathscr{U}' \subset \mathscr{U}$ of y^0 such that the following is true:*

If a $\mathscr{C}^1$ function u in $\mathscr{U}$ satisfies $Lu = 0$ in $\mathscr{U}$ and if u vanishes identically in the set:

$$\mathscr{U}^- = \{y \in \mathscr{U}; \Phi(y) < \Phi(y^0)\},$$

then $u \equiv 0$ in $\mathscr{U}'$.

The smoothness assumptions on the coefficients of L, on the function Φ and on the solution u could have been made less stringent (in particular, at the price of some extra effort, we could have dealt with $u \in H^1(\mathscr{U})$). In view of the techniques used here, the minimum assumptions on the coefficients of L and on the function Φ can be found out by inspection of the proof.

It is possible and indeed convenient to select a good coordinate system in a neighborhood of y^0. We take this point to be the origin in $\mathbb{R}^{n+1}$ (from here on we write $n = N - 1$) and denote the new coordinates by (x, t) with $x = (x^1, \ldots, x^n)$. They can be chosen in such a way that $t = 0$ is an equation of the hypersurface $\Phi(y) = \Phi(y^0)$ in an open neighborhood of y^0, which we take from here on to be Ω, and that, also in Ω, L is equal, up to a smooth nonvanishing factor, to

$$\frac{\partial}{\partial t} - i \sum_{j=1}^{n} b^j(x, t) \frac{\partial}{\partial x^j} - c(x, t), \tag{3}$$

where the vector $\mathbf{b}(x, t) = (b^1(x, t), \ldots, b^n(x, t))$ is *real*. On this refer to [2, p. 332]. From now on we assume that (3) is exactly the expression of L in the coordinates (x, t). The number z used earlier can now be taken equal to *one* and $X = \partial/\partial t$. Condition **(P)** reads:

(P)$^\sharp$ *There is an open neighborhood Ω_0 of the origin in $\mathbb{R}^n$, a number $T > 0$ and a mapping $x \mapsto \mathbf{v}(x)$ of Ω_0 into the unit sphere S^{n-1} such that*

$$\mathbf{b}(x, t) = |\mathbf{b}(x, t)| \mathbf{v}(x), \qquad \forall (x, t) \in \Omega_0 \times]-T, T[. \tag{4}$$

Let us introduce the following "critical" subset of Ω_0:

$$\mathscr{N}_0 = \{x \in \Omega_0; \forall t, |t| < T, \mathbf{b}(x, t) = 0\}. \tag{5}$$

Observe that in the complement $\Omega_0 \backslash \mathcal{N}_0$ the vector $\mathbf{v}(x)$ is unambiguously defined, and is a smooth function of x. On $\mathcal{N}_0$ we may choose it arbitrarily. We shall prove the following version of Theorem 1.

THEOREM 1′. *Suppose that* $(\mathbf{P})^s$ *holds. Let u be any function belonging to* $\mathscr{C}^1(\Omega_0 \times]-T, T[)$ *and satisfying* $Lu = 0$ *in* $\Omega_0 \times]-T, T[$. *If* $u(x, t) = 0$ *in* $\Omega_0 \times]-T, 0]$, *then* $u = 0$ *in* $\Omega_0 \times]-T, T[$.

Without embarking right away on the proof of Theorem 1′, let us indicate how it can be reduced to the case of a single "space" variable x, with

$$L = (\partial/\partial t) - ib(x, t)(\partial/\partial x) - c(x, t), \tag{6}$$

under the additional hypothesis:

$$b(x, t) \geqslant 0 \text{ everywhere.} \tag{7}$$

Indeed, let us suppose that Theorem 1′ has been proved in this particular case. Let x_0 be an arbitrary point of $\Omega_0 \backslash \mathcal{N}_0$. We may choose the coordinates $(x^1, \ldots, x^n)$ in a neighborhood of x_0, $\mathscr{W}_0$, in such a way that the vector field $V = v^1(x)(\partial/\partial x^1) + \cdots + v^n(x)(\partial/\partial x^n)$ becomes $\partial/\partial x^1$, which means that L becomes $(\partial/\partial t) - i \mid \mathbf{b}(x, t) \mid (\partial/\partial x^1) - c(x, t)$, *in* $\mathscr{W}_0$. Now $(x^2, \ldots, x^n)$ play the role of parameters. But by virtue of Theorem 1′ in the special case (6)–(7), we may conclude that $u = 0$ in $\mathscr{W}_0 \times]-T, T[$ and therefore in $(\Omega_0 \backslash \mathcal{N}_0) \times]-T, T[$. On the other hand, on any vertical segment $\{\bar{x}_0\} \times]-T, T[$ with $\bar{x}_0 \in \mathcal{N}_0$, the equation $Lu = 0$ reduces to $\partial_t u = 0$; since $u = 0$ for $t < 0$, we must have $u(\bar{x}_0, t) = 0$ for all t, $\mid t \mid < T$. This is what had to be proved.

Thus, in the forthcoming sections, we shall limit ourselves to the case of two "independent" variables (x, t) (i.e., one space variable) when (7) holds.

It is well known that there are first-order differential operators L for which uniqueness in the Cauchy problem (across a noncharacteristic hypersurface) holds without Condition $(\mathbf{P})$ being true; e.g., all the first-order operators L whose coefficients are analytic (by virtue of Holmgren's theorem). In this direction we have been able to prove the following (admittedly limited) result.

THEOREM 2. *Consider the following differential operator:*

$$L = (\partial/\partial t) - ib(x, t)(\partial/\partial x) - c(x, t),$$

whose coefficients $b(x, t)$, $c(x, t)$ *are* C^∞ *functions in an open neighborhood* Ω *of the origin in* $\mathbb{R}^2$ *(b is real).*

Suppose that $t \mapsto b(0, t)$ *vanishes of finite order at* $t = 0$. *Then, to every open neighborhood* $\mathscr{U} \subset \Omega$ *of* 0 *there is another open neighborhood* $\mathscr{U}'$ *of* 0 *such that the following is true:*

If $u \in \mathscr{C}^1(\mathscr{U})$ is such that $Lu = 0$ in $\mathscr{U}$ and $u = 0$ in $\mathscr{U}^- = \{(x, t) \in \mathscr{U}; t < 0\}$, then $u = 0$ in $\mathscr{U}'$.

Paul Cohen (ca. 1960) gave an example of a first-order linear partial differential operator, in two independent variables (with no zero-order term), for which uniqueness in the Cauchy problem across a noncharacteristic hypersurface does not hold. It can be found, as a particular case of Theorem 8.9.2 in [1].

The complexity of the situation is further underlined by the observation of C. Goulaouic that there is uniqueness in the Cauchy problem across the hyperplane $t = 0$, for all differential operators of the form

$$L = (\partial/\partial t) + ia(t)\, b(x)(\partial/\partial x), \qquad \text{in} \quad \{(x, t) \in \mathbb{R}^2; \mid x \mid < r, \mid t \mid < T\},$$

and more generally all the operators

$$L = (\partial/\partial t) + i(a_1(t)X_1 + \cdots + a_r(t)X_r) \qquad \text{in} \quad \Omega_0 \times]-T, T[,$$

where the X_j are smooth real vector fields in $\Omega_0 \subset \mathbb{R}^n$, pairwise commuting.

We wish to thank Professors C. Goulaouic and L. Nirenberg for many useful conversations on topics related to the present paper.

1. A Carleman Estimate

Let Ω be a bounded open rectangle $\mid x \mid < r$, $\mid t \mid < T$, in $\mathbb{R}^2$. We deal with two functions in Ω, $b(x, t)$, $c(x, t)$, having the following properties:

$b(x, t)$ is real-valued; (1.1)

$b(x, t)$ has distribution derivatives of order $\leqslant 2$ with respect to x which belong to $L^\infty(\Omega)$; (1.2)

$c(x, t)$ has distribution derivatives of order $\leqslant 1$ with respect to x which belong to $L^\infty(\Omega)$. (1.3)

We shall then study the first-order differential operator

$$L = \partial_t - ib(x, t)\, \partial_x - c(x, t).$$

We introduce two functions $\varphi_0(x)$, $\varphi_1(t)$ with the following properties:

φ_0 has distribution derivatives of order $\leqslant 2$ belonging to $L^\infty(-r, r)$; (1.4)

φ_1 is Lipschitz continuous in $]-T, T[$. (1.5)

We form the function

$$\varphi(x, t) = \varphi_0(x) + \int_0^t b(x, t')\, \varphi_1(t')\, dt'.$$

In view of (1.2), (1.4), and (1.5), we see that $\varphi(x, t)$ has distribution derivatives of order $\leqslant 2$ with respect to x, belonging to $L^\infty(\Omega)$.

LEMMA 1.1. *Let F be a closed subset of Ω in which $b(x, t)$ keeps the same sign (which we denote by* sgn b*). Suppose that there is a constant $c_0 > 0$ such that*

$$\{\varphi_1'(t) + (b\varphi_x)_x(x, t)\} \operatorname{sgn} b(x, t) \geqslant c_0 \text{ in } F. \tag{1.6}$$

Then, for τ_0 and C_0 positive sufficiently large, for all $\tau > \tau_0$ and all $u \in H_0^1(\Omega)$ having their support in F,

$$\tau \iint e^{2\tau\varphi} \mid u \mid^2 dx\, dt \leqslant C_0 \iint e^{2\tau\varphi}\{\mid Lu \cdot \partial_x \bar{u} \mid + \tau \mid Lu \cdot \bar{u} \mid\}\, dx\, dt. \tag{1.7}$$

Proof. We set

$$c_1(x, t) = \int_0^t c(x, s)\, ds, \qquad u = ve^{c_1}.$$

Then $Lu = e^{c_1} L_0 v$ with

$$L_0 = \partial_t - ib(\partial_x + c_{1x}).$$

Next, set $v = we^{-\tau\varphi}$. We have

$$e^{\tau\varphi} L_0 v = Mw - ibNw, \tag{1.8}$$

where

$$M = \partial_t + ib\varphi_x \tau - ibf, \tag{1.9}$$

$$N = \partial_x - i\tau\varphi_1(t) + ig, \tag{1.10}$$

having written $c_{1x}(x, t) = f + ig$, f, g real. We shall denote by (,) the inner product in $L^2_{x,t}$. Then

$$2 \operatorname{Re}(Mw - ibNw, iNw) = 2 \operatorname{Re}(Mw, iNw) - \iint b \mid Nw \mid^2 dx\, dt. \tag{1.11}$$

On the other hand,

$$2 \operatorname{Re}(w_t, iNw) = -\iint (\tau\varphi_1'(t) - g_t(x, t)) \mid w \mid^2 dx\, dt. \tag{1.12}$$

Observe that g has a first-order t-derivative which belongs to $L^\infty(\Omega)$, according to (1.3). Furthermore,

$$2\ \mathrm{Re}(ib(\varphi_x\tau - f)w, iNw) = 2\ \mathrm{Re}(b(\varphi_x\tau - f)w, (\partial_x - i\tau\varphi_1(t) + ig)w)$$

$$= -\iint (b(\tau\varphi_x - f))_x \mid w \mid^2 dx\, dt. \tag{1.13}$$

Combining (1.11), (1.12), and (1.13) yields

$$\iint b \mid Nw \mid^2 dx\, dt + \tau \iint (\varphi_1'(t) + (b\varphi_x)_x) \mid w \mid^2 dx\, dt$$

$$= -2\ \mathrm{Re}([M - ibN]w, iNw) + \iint (g_t + (bf)_x) \mid w \mid^2 dx\, dt.$$

We multiply both members by sgn b (in F) and use (1.6). Reverting from w to u easily yields (1.7), provided that

$$c_0\tau_0 > \| g_t + (bf)_x \|_{L^\infty(\Omega)}\, . \qquad \text{Q.E.D.}$$

As a conclusion to the present section we relate the validity of the estimate (1.7) to the local solvability of the pseudodifferential operator in $n + 2$ variables:

$$\tilde{L} = \tilde{M} - ib\tilde{N}, \tag{1.14}$$

where (cf. (1.9), (1.10))

$$\tilde{M} = \partial_t + ib\phi_x| D_s | - ibf, \tag{1.15}$$

$$\tilde{N} = \partial_x - i\phi_1(t) \mid D_s \mid + ig, \tag{1.16}$$

where s is the extra variable. The principal symbol of $\tilde{L}$ is given by

$$\sigma(\tilde{L}) = i(A + iB), \tag{1.17}$$

$$A = \tau + b\phi_x| \sigma |, \qquad B = -b(\xi - \phi_1(t) \mid \sigma \mid). \tag{1.18}$$

We have denoted by (ξ, τ, σ) the covariables corresponding to (x, t, s). The Hamilton–Jacobi equations for A show that the bicharacteristic strips of A can be defined by the equations

$$x = x_0\, , \qquad \xi = \xi^0 - \left\{\int_0^t (b\phi_x)_x\, dt\right\} \mid \sigma^0 \mid, \tag{1.19}$$

$$\tau = \tau^0 - (b\phi_x) \mid \sigma^0 \mid, \qquad \sigma = \sigma^0,$$

t serving as the parameter on the curve. Along such a curve,

$$B = B(x, t, \xi, \sigma) = -b(x, t) \left\{ \xi^0 - | \sigma^0 | \left(\int_0^t (b\phi_x)_x \, dt + \phi_1(t) \right) \right\}. \quad (1.20)$$

The meaning of (1.6) now becomes clear:

$$B(x, t, \xi, \sigma) = | b(x, t) | \omega(x, t, \xi^0, \sigma^0), \quad (1.21)$$

where

$$\textit{for all } (x, \xi^0, \sigma^0), \textit{ the } t\textit{-derivative of } \omega \textit{ is } \geqslant c_0 > 0. \quad (1.22)$$

This implies that $\tilde{L}$ satisfies the solvability condition (Ψ') for its transpose and thus exemplifies the general theory of [7] (see pp. 276 *et seq.*).

2. Proof of Theorem 1

As shown in the Introduction it suffices to consider the case of one space variable, when

$$L = (\partial/\partial t) - ib(x, t)(\partial/\partial x) - c(x, t), \quad (2.1)$$

under the additional hypothesis that

$$b(x, t) \geqslant 0 \textit{ in } \Omega. \quad (2.2)$$

Here Ω is an open neighborhood of the origin in $\mathbb{R}^2$, which we take to be the rectangle:

$$| x | < r, \qquad | t | < T \qquad (r, T > 0) \quad (2.3)$$

(the coefficients b and c of L are C^∞ functions in an open neighborhood of $\bar{\Omega}$). We consider a function $u \in \mathscr{C}^1(\Omega)$ which satisfies

$$Lu = 0 \text{ in } \Omega, \quad (2.4)$$

$$u(x, t) = 0 \qquad \textit{for all} \quad (x, t) \in \Omega, \quad t < 0. \quad (2.5)$$

We wish to prove that $u \equiv 0$ in Ω.

We may assume without loss of generality that $c(x, t) \equiv 0$ by utilizing the technique at the beginning of the proof of Lemma 1.1.

We shall reason by contradiction and start from the assumption that $F = \operatorname{supp} u \neq \varnothing$. Clearly we can then find a nonempty open subinterval J of $]-r, r[$ such that $u(x, t) \neq 0$ for every $x \in J$ and for some $t < T$ (depending on x). If $x \in J$ set $t(x) = \inf\{s \mid (x, s) \in F\}$; the function $t(x)$ is lower semicontinuous in J. Note that we must have $b(x, t(x)) = 0$ whatever $x \in J$. Otherwise L would be elliptic in a neighborhood of $(x, t(x))$ which belongs to the boundary ∂F, and the uniqueness in the Cauchy problem for elliptic first-order PDEs would imply that $u = 0$ in a full neighborhood of that point, which is a contradiction. For each x, $|x| < r$, let $\bar{t}(x)$ be the supremum of the numbers $t \geqslant t(x)$ such that $b(x, s) = 0$ whatever s, $t(x) \leqslant s \leqslant t$. The uniqueness in the Cauchy problem for ordinary differential equations implies that $u(x, s)$ must then also be zero for all such values of s. Since we know that $u(x, t) \neq 0$ for some $t < T$, we conclude that

$$0 \leqslant t(x) \leqslant \bar{t}(x) < T, \qquad x \in J. \tag{2.6}$$

(Note that $\bar{t}(x)$ is upper semicontinuous and that, in general, there is no reason why $t(x)$ and $\bar{t}(x)$ should be equal.) In the sequel we shall restrict our attention to the part of F which "lies above J;" i.e., the intersection of F with the open rectangle $J \times]-T, T[$, and show that it is empty, which will contradict (2.6). For the sake of simplicity, we substitute J for the interval $]-r, r[$; in fact we shall assume that $J =]-r, r[$.

Let then ρ be a number such that $0 < \rho < \inf(r, T)$ and, for all c, $0 \leqslant c \leqslant 1$, let us call E_c the set of points $(x, t) \in \mathbb{R}^2$ satisfying

$$x^2 + c(t + T)^2 < \rho^2. \tag{2.7}$$

For all $c \in [0, 1]$ E_c is contained in the vertical slab $\{(x, t); x \in J\}$. For $c = 0$, it is equal to that slab. For $c = 1$, E_c is contained in the lower half-plane $t < 0$. Let c_0 be the infimum of the numbers c such that $E_c \subset \Omega \backslash F$. Our hypothesis that F is not empty implies that $c_0 > 0$. The intersection of the boundary of E_{c_0} with that of F is not empty; let (x_0, t_0) be a point in this intersection. The normal at (x_0, t_0) to the ellipse which is the boundary of E_{c_0} is not horizontal (for this can only occur at the points $(\pm\rho, -T)$, which do not belong to F); let us call θ the angle between this normal and the vertical, i.e., a line parallel to the t-axis. We have

$$|\theta| < \pi/2. \tag{2.8}$$

We are going to show that $(x_0, t_0) \notin \operatorname{supp} u$, thus reaching a contradiction.

It is convenient to perform a translation and regard from now on (x_0, t_0) as the origin in $\mathbb{R}^2$. Let us then make the following change of variables in $\mathbb{R}^2$

$$y = x \cos\theta + t \sin\theta, \qquad s = -x \sin\theta + t \cos\theta. \tag{2.9}$$

(The angle θ has the sign which insures that the positive s-axis is the exterior normal to the above ellipse.) We have

$$L = (\cos\theta + ib \sin\theta)L', \tag{2.10}$$

where

$$L' = \frac{\partial}{\partial s} + \sin\theta\cos\theta\,\frac{1-b^2}{k^2}\,\frac{\partial}{\partial y} - i\,\frac{b}{k^2}\,\frac{\partial}{\partial y}, \tag{2.11}$$

$$k^2 = \cos^2\theta + b^2\sin^2\theta \geqslant \cos^2\theta > 0. \tag{2.12}$$

We shall then deal with L'; the trouble is that L' is not in "canonical form". Let us denote by $y(\eta, s)$ the (unique) solution of the initial value problem

$$dy/ds = \sin\theta\cos\theta((1-b^2)/k^2), \qquad y\,|_{s=0} = \eta; \tag{2.13}$$

we assume that we have replaced in $b(x, t)$ the values of x and t extracted from (2.9). We are going to apply the following general result:

LEMMA 2.1. *Let $F(y, s)$ be a real-valued $\mathscr{C}^1$ function in $\mathbb{R}^2$ whose first partial derivative with respect to y, F_y, is uniformly bounded in each slab* $\{(y, s) \in \mathbb{R}^2;\ |s| \leqslant \text{const}\}$.

Then there is a unique real $\mathscr{C}^1$ function $y = y(\eta, s)$ in $\mathbb{R}^2$ which satisfies

$$dy/ds = F(y, s), \qquad y\,|_{s=0} = \eta. \tag{2.14}$$

Furthermore, the mapping

$$(\eta, s) \to (y(\eta, s), s) \tag{2.15}$$

is a $\mathscr{C}^1$-mapping of $\mathbb{R}^2$ onto itself.

Though this lemma must be well-known, we give its proof here.

Proof. From the hypothesis we derive that, to every $a > 0$ there is $A > 0$ such that

$$|F(y_1, s) - F(y_2, s)| \leqslant A|y_1 - y_2|,$$
$$\forall y_1,\ \ y_2 \in \mathbb{R},\quad \forall s \in \mathbb{R},\quad |s| < a. \tag{2.16}$$

In view of this, Picard's iteration method shows that there is a $\mathscr{C}^1$ function of s, $|s| < a$, depending continuously on $\eta \in \mathbb{R}$, verifying (2.14). In fact it is continuously differentiable with respect to η since its partial derivative with respect to η, y_η, is the unique solution of the following initial value problem

$$dz/ds = F_y(y(\eta, s), s)z, \qquad z\,|_{s=0} = 1. \tag{2.17}$$

We see, furthermore, that

$$z = y_\eta = \exp\left(\int_0^s F_y(y(\eta, \sigma), \sigma)\,d\sigma\right). \tag{2.18}$$

Incidentally this shows that $y_n > 0$ in the whole of $\mathbb{R}^2$ and therefore that the Jacobian of (2.15) does not vanish anywhere. To prove that (2.15) is surjective is easy: η, regarded as a function of y and s, is the solution of the integral equation

$$\eta = y - \int_0^s G(\eta, \sigma)\, d\sigma, \tag{2.19}$$

with

$$G(\eta, s) = F(y(\eta, s), s). \tag{2.20}$$

By virtue of the hypotheses of F and of the formula (2.18) one sees that G_η is uniformly bounded in each slab $\{(\eta, s);\ |s| \leqslant \text{const}\}$. Consequently, Picard's method shows that (2.19) has a unique solution for all $(y, s) \in \mathbb{R}^2$. This completes the proof of Lemma 2.1.

We apply Lemma 2.1 to Problem (2.13). We may assume that $b(x, t) \equiv 0$ outside a sufficiently large bounded subset of $\mathbb{R}^2$ (whose interior contains the closure of Ω); this insures that the hypotheses of Lemma 2.1 are fulfilled, and therefore, that the diffeomorphism (2.15) is global. We now find ourselves in the following situation: L', in the new coordinates η, s, has the canonical form

$$L' = (\partial/\partial s) - i\beta(\eta, s)(\partial/\partial \eta), \qquad \beta \geqslant 0 \text{ in } \Omega. \tag{2.21}$$

We know that $L'u = 0$ in Ω and that $u = 0$ in the set defined, in the "old" coordinates by $x^2 + c_0(t + T)^2 < \rho^2$. But there is a very important new feature in the situation which can be described as follows. Let T' be the least number > 0 such that the point $(0, T')$ belongs to the complement of Ω. Then,

$$\textit{for some } s_0,\ 0 < s_0 < T', \qquad \beta(0, s_0) > 0. \tag{2.22}$$

Indeed, if this were not true, β would vanish on the whole segment of the s-axis joining the origin to the boundary of Ω. In other words, the imaginary part of the symbol of L' would vanish identically on the corresponding arc of (null) bicharacteristic strip of the real part. But such a property, in conjunction with (**P**), is invariant under coordinate changes and under multiplication of L by a nonvanishing complex (smooth) function (for a proof, see [6, Introduction]. Therefore, the original coefficient $b(x, t)$ should also have vanished identically on the vertical segment joining (x_0, t_0) to the boundary of Ω, that is to say, to (x_0, T). This would mean that $\bar{t}(x_0) = T$ contrary to (2.6). This proves (2.22).

Let us now select a number $K > 0$ sufficiently large so that the parabola $s + (1/2)K\eta^2 = 0$ is completely contained (except for its vertex, the origin) in the interior of the region where u vanishes identically (at least in some neighborhood of the origin) and let us make the change of variables

$$\tau = s + (1/2)K\eta^2, \qquad \xi = \eta. \tag{2.23}$$

The expression of L' in the new coordinates is

$$L' = (1 - iK\xi\beta)L'', \tag{2.24}$$

with

$$L'' = \frac{\partial}{\partial\tau} + K\xi\frac{\beta^2}{\kappa^2}\frac{\partial}{\partial\xi} - i\frac{\beta}{\kappa^2}\frac{\partial}{\partial\xi}. \tag{2.25}$$

Here,

$$\kappa^2 = 1 + K^2\xi^2\beta^2. \tag{2.26}$$

Once more we must put the operator under study, now L'', into canonical form. We solve the initial value problem:

$$d\xi/d\tau = K\xi\beta^2/\kappa^2, \qquad \xi\,|_{\tau=0} = x. \tag{2.27}$$

Let $\Phi(\xi, \tau)$ denote the right-hand side in the ODE in (2.27). We have:

$$\Phi_\xi = K\frac{\beta^2}{\kappa^2} + \frac{K\xi\beta}{\kappa^2}\beta_\xi - 2\frac{K\xi\beta^2}{\kappa^4}(K^2\xi^2\beta\beta_\xi + K^2\xi\beta^2),$$

whence, in view of (2.26),

$$|\,\Phi_\xi\,| \leqslant (3/2)(K\beta^2 + |\,\beta_\xi\,|). \tag{2.28}$$

Since the support of β, which is equal to that of b, is compact, we reach the conclusion that Lemma 2.1 is applicable to (2.27) and that the solution $\xi = \xi(x, \tau)$ gives rise to a global $\mathscr{C}^1$-mapping of $\mathbb{R}^2$ onto itself,

$$(x, t) \mapsto (\xi(x, t), t). \tag{2.29}$$

We may now write

$$L'' = (\partial/\partial t) - iB(x, t)(\partial/\partial x), \qquad B \geqslant 0 \text{ in } \Omega. \tag{2.30}$$

It should be kept in mind that the new coordinates (x, t) are different from the original ones, but these we shall not use any more and no confusion need be feared. Observe that, because of the presence of the factor ξ in $\Phi(\xi, \tau)$, the unique solution of (2.27) when $x = 0$ is $\xi = 0$. This means that the diffeomorphism (2.29) not only preserves the x-axis, but also preserves the t-axis. Since $B = \beta/\kappa^2$, we see that there is a number $t_0 > 0$ such that $B(0, t_0) > 0$ and such that the closed segment joining the origin to $(0, t_0)$ is contained in Ω. The useful feature in the present situation is that not only does u vanish for $t < 0$, but that, in fact, u vanishes for $t < \gamma x^2$ for a suitable $\gamma > 0$. In other words, we are in the standard situation for applying the Carleman estimates: the support of the solution is contained in a strictly convex subset of the region on one side of the noncharacteristic surface under study.

Let us introduce the largest number $t_1 \geqslant 0$ such that $B(0, s) = 0$ for all s, $0 \leqslant s \leqslant t_1$; of course we have $t_1 < t_0$. We shall now reason in a rectangle

$$\Omega' = \{(x, t); |x| < \epsilon, -\epsilon < t < t_1 + \epsilon\}, \tag{2.31}$$

where ϵ is a sufficiently small positive number, in order that a number of requirements be fulfilled: (1) the closure of Ω' must be contained in Ω; (2) u must vanish identically in the subset of Ω' defined by $t < \gamma x^2$; (3) $|B| + |B_x| < \delta$ in Ω', for a suitably small $\delta > 0$ (since $B \geqslant 0$ everywhere, B_x must vanish wherever B vanishes, in particular on the segment joining the origin to $(0, t_1)$). The last step in the proof will be to apply Lemma 1.1 with L'' in the place of L and Ω' in that of Ω. We show how to choose the function ϕ in the exponential of the Carleman estimate (1.7). We take

$$\phi_0(x) = -x^2, \qquad \phi_1(t) = -e^{-t},$$

so that

$$\phi(x, t) = -x^2 - \int_0^t B(x, s)\, e^{-s}\, ds.$$

On one hand we have, in Ω',

$$\phi_1' \geqslant e^{-(t_1+\epsilon)},$$

and on the other,

$$\sup|(B\phi_x)_x| \leqslant C \sup(|B| + |B_x|) \leqslant C\delta.$$

Thus the hypothesis (1.6) is satisfied, as soon as ϵ, and therefore also δ, are sufficiently small. We conclude that the inequality (1.7), where L'' replaces L, is valid (here u is an arbitrary element of $H_0^1(\Omega')$ and not the solution under study). We apply Theorem 2.3 of [4] (rather, an obvious modification of it) and conclude that our solution u of $Lu = 0$ must vanish identically in a neighborhood of the intersection

$$\{(x, t); t \geqslant \gamma x^2\} \cap \{(x, t) \in \Omega'; \phi(x, t) \geqslant 0\}, \tag{2.32}$$

provided this intersection is compact. Since

$$\phi(x, t) = -x^2 - \int_0^t B(x, s)\, e^{-s}\, ds,$$

the set (2.32) is exactly equal to the closed segment

$$x = 0, \qquad 0 \leqslant t \leqslant t_1, \tag{2.33}$$

which is indeed a compact subset of Ω'. In particular we see that u must vanish in a neighborhood of the origin, contrary to our initial assumption.

3. Proof of Theorem 2

In this section we suppose that $t \mapsto b(0, t)$ vanishes of finite order k at $t = 0$ (when k is *odd*, the operator is not locally solvable at the origin). According to the Weierstrass–Malgrange preparation theorem, we may write, in a neighborhood of the origin which we take to be $\Omega = \{(x, t);\ |x| < r,\ |t| < T\}$,

$$b(x, t) = E(x, t)(t^k + a_1(x)t^{k-1} + \cdots + a_k(x)), \tag{3.1}$$

where E does not vanish anywhere in Ω, whereas all the $a_j(0)$ are zero. Let us set

$$f(x, t) = t^k + a_1(x)t^{k-1} + \cdots + a_k(x).$$

Lemma 3.1. *There are k open subsets $\mathcal{O}_1, \ldots, \mathcal{O}_k$ of the interval $|x| < r$, whose union is dense in this interval, and such that*

for each $j = 1, \ldots, k$, and every $x \in \mathcal{O}_j$ the polynomial in the variable t, $f(x, t)$, has exactly j distinct roots. (3.2)

Proof. For each x, $|x| < r$, let $d(x)$ be the number of distinct roots of $f(x, t)$; $d(x)$ is an upper-semicontinuous function in $]-r, r[$, and takes its values in the set of integers $1, \ldots, k$. The set where $d(x) = k$ is therefore open; we take it to be $\mathcal{O}_k$. Let $\mathcal{O}_k'$ be the interior of the complement of $\mathcal{O}_k$ in $]-r, r[$; the subset of $\mathcal{O}_k'$ in which $d(x) = k - 1$ is an open subset of $\mathcal{O}_k'$ which we take to be $\mathcal{O}_{k-1}$. We call $\mathcal{O}_{k-1}'$ the interior of the complement of $\mathcal{O}_{k-1}$ in $\mathcal{O}_k'$ and we call $\mathcal{O}_{k-2}$ the subset of $\mathcal{O}_{k-1}'$ in which $d(x) = k - 2$, etc. The open sets $\mathcal{O}_j$ thus defined fulfill the requirements of the lemma.

Observe that the roots of $f(x, t)$ can be represented as *continuous* functions of x in the whole interval $|x| < r$. If the multiplicity of a given root $\rho(x)$ remains constant, say equal to d, in some open subset U of that interval, then $\rho(x)$ is a simple root of the polynomial in t,

$$(\partial/\partial t)^{d-1} f(x, t).$$

By the implicit function theorem we derive that $\rho(x)$ is a C^∞ function of x in U. Observe also that if $\rho(x)$ is nonreal for points x arbitrarily close to x_0 but is real for $x = x_0$, the same is then true of its complex conjugate, which is also a root of the real polynomial $f(x, t)$. It follows from this that the multiplicity of $\rho(x)$ must increase by at least one unit at $x = x_0$. Let us summarize.

LEMMA 3.2. *Let $\mathcal{O}_j$ $(j = 1,..., k)$ be the open sets in Lemma* 3.1.

In each $\mathcal{O}_j$ the distinct roots of $f(x, t)$ can be represented by j C^∞ functions $\rho_{j'}(x)$; (3.3)

whatever $j = 1,..., k$ and $j' = 1,..., j$, if $\rho_{j'}(x)$ is real at some point x_0 of $\mathcal{O}_j$, it is real throughout the connected component of x_0 in $\mathcal{O}_j$. (3.4)

Proof of Theorem 2. It will suffice to show that $u = 0$ in $U \times]-T, T[$, where U is any connected component of any one of the $\mathcal{O}_j$. Let

$$\rho_1 < \cdots < \rho_\ell \qquad (\ell \leqslant j)$$

denote those, among the distinct roots of $f(x, t)$ for x in $\mathcal{O}_j$, which are real in U. Let us call C_i the curve defined by the equation

$$t = \sup(-T, \inf(\rho_i(x), T)), \qquad x \in U \tag{3.5}$$

$(i = 1,..., \ell)$. Let us call A_0 the open subset of $U \times]-T, T[$ which lies below C_1, A_ℓ the one which lies above C_ℓ, A_i $(i = 1,..., \ell - 1)$ the one which lies between C_i and C_{i+1}. Observe that the differential operator L is elliptic in each open set A_i: thus if u vanishes in some open subset of A_i, it vanishes throughout A_i (which is clearly connected). Let i_0 be the least i such that $A_i \neq \varnothing$ and u does not vanish identically in A_i (we reason by contradiction and are going to show that such an index i_0 cannot exist). Then clearly A_{i_0} cannot intersect the lower region $U \times]-T, 0[$, where u vanishes. Consequently, the curve C_{i_0} must lie entirely in the region $x \in U$, $t \geqslant 0$; and if A_{i_0} is not to be empty, it must also lie below the segment $x \in U$, $t = T$; hence there must be points x in U where the equation of C_{i_0} is $t = \rho_{i_0}(x) < T$. Above such points (which form an open set) C_{i_0} is a smooth noncharacteristic curve; u vanishes identically below it, whereas $b(x, t)$ does not change sign in A_{i_0}. We may apply Theorem 1′ and conclude that $u = 0$ in a neighborhood of $C_{i_0} \cap (U \times]-T, T[)$, hence in the whole of A_{i_0} by ellipticity: hence a contradiction.

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